NORMALITY OF DILATED POLYTOPES

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ABSTRACT. Let $\mathcal{P}$ be an integral convex polytope of dimension d and $n\mathcal{P}$, where $n = 1, 2, \ldots$, dilated polytopes of $\mathcal{P}$. It is natural to ask, for which integers $q > 0$, the dilated polytope $q\mathcal{P}$ is normal. Let $\mu(\mathcal{P})$ denote the maximal degree of the Hilbert basis of the polyhedral cone arising from $\mathcal{P}$. It is known that $\mu(\mathcal{P}) \leq d-1$ and that $q\mathcal{P}$ is normal for all $q \geq \mu(\mathcal{P})$. In this paper, it is proved that, given an integer $d \geq 4$, there exists an integral convex polytope $\mathcal{P}$ of dimension d with $\mu(\mathcal{P}) = d-1$ such that $(d-2)\mathcal{P}$ is normal. Moreover, given integers $d \geq 3$ and $2 \leq j \leq d-1$, we show the existence of an empty simplex $\mathcal{P}$ of dimension d with $j = \mu(\mathcal{P})$ such that $q\mathcal{P}$ cannot be normal for any $1 \leq q < j$.

INTRODUCTION

The integral convex polytope has been studied from viewpoints of commutative algebra and algebraic geometry together with enumerative combinatorics. Recall that an *integral* convex polytope is a convex polytope all of whose vertices have integer coordinates. We say that an integral convex polytope $\mathcal{P} \subset \mathbb{R}^N$ is *normal* (or possesses the *integer decomposition property*) if, for any integer $n = 1, 2, \ldots$ and for any $\alpha \in n\mathcal{P} \cap \mathbb{Z}^N$, where $n\mathcal{P}$ is the dilated polytope $\{n\alpha : \alpha \in \mathcal{P}\} \subset \mathbb{R}^N$ of $\mathcal{P}$, there exist $\alpha_1, \ldots, \alpha_n$ belonging to $\mathcal{P} \cap \mathbb{Z}^N$ such that $\alpha = \alpha_1 + \cdots + \alpha_n$.

Let $\mathcal{C} \subset \mathbb{R}^N$ be a *pointed, rational and polyhedral cone* generated by rational vectors $\mathbf{a}_1, \ldots, \mathbf{a}_m \in \mathbb{Q}^N$. Thus

$$\mathcal{C} = \{\lambda_1 \mathbf{a}_1 + \cdots + \lambda_m \mathbf{a}_m : \lambda_1, \ldots, \lambda_m \in \mathbb{R}_{\geq 0}\}$$

such that $\{0\}$ is the largest linear subspace contained in $\mathcal{C}$. A finite set of integer vectors $\{\mathbf{h}_1, \ldots, \mathbf{h}_s\} \subset \mathbb{Z}^N$ is called a *Hilbert basis* of $\mathcal{C}$ if

$$\mathcal{C} \cap \mathbb{Z}^N = \{\lambda_1 \mathbf{h}_1 + \cdots + \lambda_s \mathbf{h}_s : \lambda_1, \ldots, \lambda_s \in \mathbb{Z}_{\geq 0}\}.$$

A Hilbert basis exists (Gordan; 1873) and a minimal Hilbert basis is unique (van der Corput; 1931). Let $\mathcal{H}(\mathcal{C})$ denote *the* minimal Hilbert basis of $\mathcal{C}$.

Given an integral convex polytope $\mathcal{P} \subset \mathbb{R}^N$, we write $\widetilde{\mathcal{P}} \subset \mathbb{R}^{N+1}$ for the integral convex polytope $\{(\alpha, 1) \in \mathbb{R}^{N+1} : \alpha \in \mathcal{P}\}$. Let $\mathcal{C}(\mathcal{P}) \subset \mathbb{R}^{N+1}$ denote the pointed, rational and polyhedral cone generated by those vectors $(\alpha, 1) \in \widetilde{\mathcal{P}} \cap \mathbb{Z}^{N+1}$ such that α is a vertex of $\mathcal{P}$. The *degree* of $(\alpha, n) \in \mathcal{C}(\mathcal{P}) \cap \mathbb{Z}^{N+1}$ is $\deg(\alpha, n) = n$.

2010 Mathematics Subject Classification: Primary 52B20; Secondary 14Q15.
Keywords: normal polytope, integer decomposition property, dilated polytope, Hilbert basis.

Let K be a field and $S = K[x_1, x_1^{-1}, \ldots, x_N, x_N^{-1}, t]$ the Laurent polynomial ring in $N+1$ variables over K. One can associate each $(\mathbf{a}, n) = (a_1, \ldots, a_N, n) \in \mathbb{Z}^{N+1}$, where $n = 0, 1, 2, \ldots$, with the Laurent monomial $\mathbf{x}^{\mathbf{a}} t^n = x_1^{a_1} \cdots x_N^{a_N} t^n$ of S.

The *toric ring* of an integral convex polytope $\mathcal{P} \subset \mathbb{R}^N$ is the subring $K[\mathcal{P}] \subset S$ generated by those monomials $\mathbf{x}^{\mathbf{a}} t$ with $(\mathbf{a}, 1) \in \mathcal{A}_{\mathcal{P}}$. The *Ehrhart ring* of $\mathcal{P}$ is the subring $\mathcal{E}_K(\mathcal{P}) \subset S$ generated by those monomials $\mathbf{x}^{\mathbf{a}} t^n$ with $(\mathbf{a}, n) \in \mathcal{H}(\mathcal{C}(\mathcal{P}))$.

It then follows immediately that, given an integral convex polytope $\mathcal{P} \subset \mathbb{R}^N$, the following conditions are equivalent:

- $\mathcal{P}$ is normal;
- $K[\mathcal{P}] = \mathcal{E}_K(\mathcal{P})$;
- Each element of $\mathcal{H}(\mathcal{C}(\mathcal{P}))$ is of degree 1.

Our motivation to organize the present paper is to discuss the question that, for which $q > 0$, the dilated polytope $q\mathcal{P}$ is normal.

Let $\mathcal{P} \subset \mathbb{R}^N$ be an integral convex polytope of dimension d. Write $\mu(\mathcal{P})$ for the maximal degree of the elements belonging to $\mathcal{H}(\mathcal{C}(\mathcal{P}))$. It is known [1, Lemma 2.2.16] that $\mu(\mathcal{P}) \leq d-1$ and that the dilated polytope $q\mathcal{P}$ is normal for all $q \geq \mu(\mathcal{P})$. Thus it would be of interest to find the set $\mathcal{N}_{\mathcal{P}}$ of those integers $q \in \mathbb{Z}$ with $0 < q < \mu(\mathcal{P})$ such that $q\mathcal{P}$ is normal.

A simple observation on $\mathcal{N}_{\mathcal{P}}$ is

Lemma 0.1. *Each $q \in \mathcal{N}_{\mathcal{P}}$ cannot divide $\mu(\mathcal{P})$.*

Proof. Let $q \in \mathcal{N}_{\mathcal{P}}$ which divides $\mu(\mathcal{P})$. Let $(\alpha, j_0) \in \mathcal{H}(\mathcal{C}(\mathcal{P}))$ with $j_0 = \mu(\mathcal{P})$. Then $\alpha \in n(q\mathcal{P}) \cap \mathbb{Z}^N$, where $n = j_0/q$. Since $q\mathcal{P}$ is normal, one has $\alpha = \alpha_1 + \cdots + \alpha_n$ with each $\alpha_i \in q\mathcal{P} \cap \mathbb{Z}^N$. Hence $(\alpha, j_0) = (\alpha_1, q) + \cdots + (\alpha_n, q)$. This contradicts the fact that $(\alpha, j_0) \in \mathcal{H}(\mathcal{C}(\mathcal{P}))$. Hence q cannot divide $\mu(\mathcal{P})$, as required. $\square$

One of the research problems is the following

Problem 0.2. Given arbitrary integers $1 \leq q \leq j \leq d-1$, construct an integral convex polytope $\mathcal{P}$ of dimension d such that

- $j = \mu(\mathcal{P})$;
- $q\mathcal{P}$ is normal;
- $r\mathcal{P}$ cannot be normal for all $1 \leq r < q$.

It seems to be reasonable to ask the following

Question 0.3. Let $\mathcal{P}$ be an integral convex polytope and suppose that $q\mathcal{P}$ is normal. Then is $r\mathcal{P}$ normal for all $r > q$?

In the present paper, two partial answers for Problem 0.2 will be given. First, Theorem 0.4 will be proved in Section 1. Second, a proof of Theorem 0.5 will be given in Section 2. Even though the answer of Question 0.3 seems to be negative, we do not succeed in discovering any counterexample.

Theorem 0.4. *Given an integer $d \geq 4$, there exists an integral convex polytope $\mathcal{P}$ of dimension d with $\mu(\mathcal{P}) = d-1$ such that $(d-2)\mathcal{P}$ is normal.*

Recall that an *empty simplex* is an integral simplex which possesses no integer points except for its vertices.

Theorem 0.5. *Given integers $d \geq 3$ and $2 \leq j \leq d-1$, there exists an empty simplex $\mathcal{P}$ of dimension d with $j = \mu(\mathcal{P})$ such that $q\mathcal{P}$ cannot be normal for all $1 \leq q < j$.*

1. Proof of Theorem 0.4

Let $\mathbf{e}_i$ be the ith unit coordinate vector of $\mathbb{R}^d$ and $\mathbf{0} = (0, \ldots, 0) \in \mathbb{R}^d$. Work with a fixed integer $d \geq 4$ and $M = d(d-2)+1$. Define $v_j \in \mathbb{Z}^d$, $1 \leq i \leq d$, as follows:

$$v_j = \begin{cases} \mathbf{0}, & j = 0, \\ \mathbf{e}_j, & j = 1, \ldots, d-1, \\ \mathbf{e}_1 + \cdots + \mathbf{e}_{d-1} + M\mathbf{e}_d, & j = d. \end{cases}$$

Let $v'_j = v_j + \mathbf{e}_d$ for $j = 0, 1, \ldots, d$. Write $\mathcal{P} \subset \mathbb{R}^d$ for the integral convex polytope of dimension d with the vertices $v_0, v_1, \ldots, v_d$ and $v'_0, v'_1, \ldots, v'_d$. Such the convex polytope appears in [3, Section 1].

It will be proved that $\mathcal{P}$ enjoys the required properties of Theorem 0.4, i.e., $\mu(\mathcal{P}) = d-1$ and $(d-2)\mathcal{P}$ is normal. It follows that $\mathcal{P}$ contains no integer point in its interior.

(First Step) First of all, the minimal Hilbert basis $\mathcal{H}(\mathcal{C}(\mathcal{P}))$ must be computed. If $q \in \{1, \ldots, M-1\}$, then there exist unique integers k and s with $1 \leq k \leq d-2$ and $1 \leq s \leq d$ with $q = (k-1)d + s$. Since

$$\begin{aligned} \frac{(d-2)s-k+1}{M}(v_0, 1) + \frac{M-q}{M}\sum_{j=1}^{d-1}(v_j, 1) + \frac{q}{M}(v_d, 1) \\ = (\mathbf{e}_1 + \cdots + \mathbf{e}_{d-1} + q\mathbf{e}_d, d-k) \in \mathbb{Z}^{d+1}, \end{aligned}$$

it follows that $(\mathbf{e}_1 + \cdots + \mathbf{e}_{d-1} + q\mathbf{e}_d, d-k) \in \mathcal{C}(\mathcal{P}) \cap \mathbb{Z}^{d+1}$. When $k = 1$ and $s = 1, \ldots, d-1$, one has $q = s$ and

$$(\mathbf{e}_1 + \cdots + \mathbf{e}_{d-1} + q\mathbf{e}_d, d-1) = \sum_{j=1}^{s}(v'_j, 1) + \sum_{j=s+1}^{d-1}(v_j, 1).$$

Hence $(\mathbf{e}_1 + \cdots + \mathbf{e}_{d-1} + q\mathbf{e}_d, d-1)$ cannot belong to $\mathcal{H}(\mathcal{C}(\mathcal{P}))$. Now, it is routine work to show that, by considering the facets of the cone $\mathcal{C}(\mathcal{P})$, the minimal Hilbert basis $\mathcal{H}(\mathcal{C}(\mathcal{P}))$ coincides with

$$(\widetilde{\mathcal{P}} \cap \mathbb{Z}^{d+1}) \cup \{(\mathbf{e}_1 + \cdots + \mathbf{e}_{d-1} + q\mathbf{e}_d, d - \lfloor (q-1)/d \rfloor - 1) : q = d, \ldots, M-1\}.$$

Thus in particular $\mu(\mathcal{P}) = d-1$.

(Second Step) Let

$$u_s^{(k)} = \mathbf{e}_1 + \cdots + \mathbf{e}_{d-1} + ((k-1)d + s)\mathbf{e}_d,$$

where $k = 2, \ldots, d-2$ and $s = 1, \ldots, d$, and

$$u = \mathbf{e}_1 + \cdots + \mathbf{e}_{d-1} + d\mathbf{e}_d.$$

One can easily see the identities

$$(u, d-1) + (v_i, 1) = (v_i', 1) + \sum_{j=1}^{d-1} (v_j', 1) \ \text{ for } \ i = 0, 1, \ldots, d,$$

$$(u, d-1) + (v_i', 1) = (v_0, 1) + (v_i, 1) + (u_1^{(2)}, d-2) \ \text{ for } \ i = 0, 1, \ldots, d,$$

$$(u, d-1) + (u_s^{(d-2)}, 2) = (v_0, 1) + (v_d, 1) + \sum_{j=1}^{s-1} (v_j', 1) + \sum_{j=s}^{d-1} (v_j, 1),$$

$$(u, d-1) + (u_s^{(k)}, d-k) = (u_s^{(k+1)}, d-k-1) + \sum_{j=0}^{d-1} (v_j, 1) \ \text{ for } \ k = 2, \ldots, d-3,$$

$$(u, d-1) + (u, d-1) = (u_d^{(2)}, d-2) + \sum_{j=0}^{d-1} (v_j, 1).$$

It then follows that

$$(\mathcal{C}(\mathcal{P}) \cap \mathbb{Z}^{d+1}) \setminus \{(u, d-1)\} = \mathbb{Z}_{\geq 0}(\mathcal{H}(\mathcal{C}(\mathcal{P})) \setminus \{(u, d-1)\}).$$

Moreover, if $k + k' \geq d$, then

$$(u_s^{(k)}, d-k) + (u_{s'}^{(k')}, d-k') = \begin{cases} (v_d, 1) + (u_{s+s'-1}^{(k+k'-d+1)}, 2d-k-k'-1), & \text{if } \ s + s' \leq d+1, \\ (v_0, 1) + (v_d, 1) + (u_{s+s'-1-d}^{(k+k'-d+2)}, 2d-k-k'-2), & \text{if } \ s + s' \geq d+2. \end{cases}$$

If $k + k' \leq d-1$, then

$$(u_s^{(k)}, d-k) + (u_{s'}^{(k')}, d-k') = \begin{cases} (u_s^{(k+k'-1)}, d-k-k'+1) + \sum_{j=1}^{s'} (v_j', 1) + \sum_{j=s'+1}^{d-1} (v_j, 1), & \text{if } \ s' \leq d-1, \\ (u_s^{(k+k')}, d-k-k') + \sum_{j=0}^{d-1} (v_j, 1), & \text{if } \ s' = d. \end{cases}$$

Consequently, when we write $\alpha \in \mathcal{C}(\mathcal{P}) \cap \mathbb{Z}^{d+1}$ by using more than one $u_s^{(k)}$'s, we can reduce one $u_s^{(k)}$. Hence α can be expressed by using at most one $u_s^{(k)}$ belonging to $\mathcal{H}(\mathcal{C}(\mathcal{P}))$.

(Third Step) Let $\mathcal{P}' = (d-2)\mathcal{P}$ and $\alpha \in n\mathcal{P}' \cap \mathbb{Z}^d$. Since $d \geq 4$, one has $n(d-2) \neq d-1$. Thus $\alpha \neq u$. By virture of the (Second Step), there exists an

expression of α of the form

$$(\alpha, n(d-2)) = v' + \sum_{j=1}^{n(d-2)-\deg(v')} (v''_j, 1),$$

where $v' \in \{(u_s^{(k)}, d-k) : k = 2, \ldots, d-2, s = 1, \ldots, d\}$ and each $v''_j \in \mathcal{P} \cap \mathbb{Z}^d$. Since the degree of v' is at most $d-2$, there exists an expression of α of the form

$$\alpha = \alpha_1 + \cdots + \alpha_n$$

with each $\alpha_i \in \mathcal{P}' \cap \mathbb{Z}^d$. Hence $\mathcal{P}'$ is normal, as desired.

2. Proof of Theorem 0.5

Fix positive integers d and j with $d \geq 3$ and $2 \leq j \leq d-1$. Write $\mathcal{P}(d, j) \subset \mathbb{R}^d$ for the convex hull of $\{w_0, w_1, \ldots, w_d\} \subset \mathbb{Z}^d$, where

$$w_i = \begin{cases} \mathbf{0}, & i = 0, \\ \mathbf{e}_i, & i = 1, \ldots, d-1, \\ \mathbf{e}_1 + \mathbf{e}_2 + \cdots + \mathbf{e}_j + j\mathbf{e}_d, & i = d. \end{cases}$$

Thus $\mathcal{P}(d, j)$ is an integral simplex of dimension d whose vertices are $\mathbf{0}$ together with the row vectors of the $d \times d$ matrix

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & 1 & \ddots & & & & 0 \\ \vdots & \ddots & \ddots & \ddots & & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & & \ddots & \ddots & \ddots & \vdots \\ 0 & & & & \ddots & 1 & 0 \\ 1 & \cdots & 1 & 0 & \cdots & 0 & j \end{pmatrix}, \tag{1}$$

where there are j 1's in the dth row.

Let $\mathcal{Q} = \mathcal{P}(d, j)$ and $(\delta_0, \delta_1, \ldots, \delta_d)$ the δ-*vector* of $\mathcal{Q}$. The purpose of this section is to prove that $\mathcal{Q}$ enjoys the required properties of Theorem 0.5. Consult, e.g., [2, Part II] for fundamental materials on δ-vectors of integral convex polytopes.

Since the determinant of (1) is j, the normalized volume of $\mathcal{Q}$ is j. Moreover,

$$\frac{q}{j} w_0 + \frac{q}{j} \sum_{i=1}^{j} (w_i, 1) + \frac{j-q}{j} (w_d, 1) = (\mathbf{e}_1 + \cdots + \mathbf{e}_j + (j-q)\mathbf{e}_d, q+1), \tag{2}$$

where $q = 1, \ldots, j-1$. Thus $\delta_q \geq 1$ for $q = 2, \ldots, j$ ([2, Proposition 27.7]). Since $\sum_{i=0}^{d} \delta_i = j$, one has

$$(\delta_0, \delta_1, \ldots, \delta_d) = (1, 0, \underbrace{1, \ldots, 1}_{j-1}, 0, \ldots, 0).$$

In particular $\delta_1 = 0$ and $\mathcal{Q}$ is an empty simplex. It follows from (2) that the Hilbert basis $\mathcal{H}(\mathcal{C}(\mathcal{Q}))$ is

$$(\widetilde{\mathcal{Q}} \cap \mathbb{Z}^{d+1}) \cup \{(\mathbf{e}_1 + \cdots + \mathbf{e}_j + (j - q + 1)\mathbf{e}_d, q) : q = 2, \ldots, j\}.$$

Hence, $\mu(\mathcal{Q}) = j$.

Now, we show that $r\mathcal{Q}$ cannot be normal for $1 \leq r < j$. Clearly, $\mathcal{Q}$ cannot be normal. Let r be an integer with $2 \leq r < j$ and m the least common multiple of r and j. Write $m = rg$ with $g \geq 2$. Let

$$\alpha = ((m - j + 1)\mathbf{e}_1 + \mathbf{e}_2 + \cdots + \mathbf{e}_j + \mathbf{e}_d, m) \in \mathbb{Z}^{d+1}.$$

Since

$$\alpha = (m - j)(w_1, 1) + (\mathbf{e}_1 + \cdots + \mathbf{e}_j + \mathbf{e}_d, j),$$

it follows that α belongs to $\mathcal{C}(\mathcal{Q}) \cap \mathbb{Z}^{d+1}$ and its degree is m. Hence α belongs to $\mathcal{C}(r\mathcal{Q}) \cap \mathbb{Z}^{d+1}$ and its degree is g. One has $\alpha \notin \mathbb{Z}_{\geq 0}(\widetilde{\mathcal{Q}} \cap \mathbb{Z}^{d+1})$. Suppose that $\alpha \in \mathbb{Z}_{\geq 0}(\widetilde{r\mathcal{Q}} \cap \mathbb{Z}^{d+1})$ with $\alpha = (\alpha_1, 1) + \cdots + (\alpha_g, 1)$, where $\alpha_1, \ldots, \alpha_g \in r\mathcal{Q} \cap \mathbb{Z}^d$. Then the dth coordinate of each of $\alpha_1, \ldots, \alpha_g$ must be 0 or 1. Since each $\beta \in \mathcal{H}(\mathcal{C}(\mathcal{Q}))$ with $2 \leq \deg \beta \leq r$ is of the form

$$\beta = (\mathbf{e}_1 + \cdots + \mathbf{e}_j + (j - i + 1)\mathbf{e}_d, i),$$

where $i = 2, \ldots, r$, none of $\alpha_1, \ldots, \alpha_g$ can be expressed by using such elements. Thus each of $\alpha_1, \ldots, \alpha_g$ must be written as the sum of r elements belonging to $\widetilde{\mathcal{Q}} \cap \mathbb{Z}^{d+1}$. It then follows that α can be written as the sum of m elements belonging to $\widetilde{\mathcal{Q}} \cap \mathbb{Z}^{d+1}$. This contradicts $\alpha \notin \mathbb{Z}_{\geq 0}(\widetilde{\mathcal{Q}} \cap \mathbb{Z}^{d+1})$. Hence $\alpha \notin \mathbb{Z}_{\geq 0}(\widetilde{r\mathcal{Q}} \cap \mathbb{Z}^{d+1})$. Thus there exists $\gamma \in \mathcal{H}(\mathcal{C}(r\mathcal{Q}))$ with $\deg \gamma \geq 2$. Consequently, $r\mathcal{P}$ cannot be normal, as required.

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